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Principles of Economics and Management I

Session 3 · Demand and Elasticity

Introduction



Where We Came From

Session 2 ended holding a **function**: the Marshallian demand $x^*(p_x, p_y, M)$, solved out of one consumer's problem.

One consumer is not a market. Today we add them up, and then we **measure** how sharply the total responds. 

Elasticity is that measure. For anyone coming from mathematics, it is a **normalized derivative**.



Today's Map

1. ✚ From one consumer's demand to **market** demand
2. 💎 Value, reservation prices and **consumer surplus**
3. 📐 Price elasticity: the point definition and the three regimes
4. 📈 Polar cases of elasticity
5. 💰 Elasticity and **revenue**, and the link to marginal revenue
6. 💰 Income and cross-price elasticity, and all three in logs

We finish with the tool that session 4 needs to talk about market power. 🔧

Part 1 · From the Individual to the Market

One Consumer's Demand

Fix p_y and M , let p_x vary, and last session's solution traces a curve.

For Cobb-Douglas, $x^* = \frac{\alpha M}{p_x}$, so $\frac{\partial x^*}{\partial p_x} < 0$. Downward sloping, and **derived**, not assumed.

Economists plot it with p_x on the **vertical** axis, which is the inverse function. A historical accident you simply have to live with.



+ From One Consumer to the Market

At each price, ask every consumer how much they want and **add the quantities**.

$$Q(p) = \sum_{i=1}^n x_i^*(p, \cdot)$$

It is a sum **across quantities at a given price**, so on a diagram you add **horizontally**, not vertically.

Each individual demand slopes down, so the sum does too. Market demand inherits the property from the consumer's problem.

Value First, Price Second

Now a second route to the same curve, one that makes surplus visible. Ana is thinking about buying a concert ticket.

Value v : what the ticket is worth **to Ana**, measured in euros. It is a property of Ana and the good, not of the market.

Ana's value is **one number**. It does not move when the price moves. What the price decides is only whether buying is worth it.

The Buyer's Reservation Price

Ana buys when her **surplus** is not negative:

$$\text{surplus} = v - p \geq 0$$

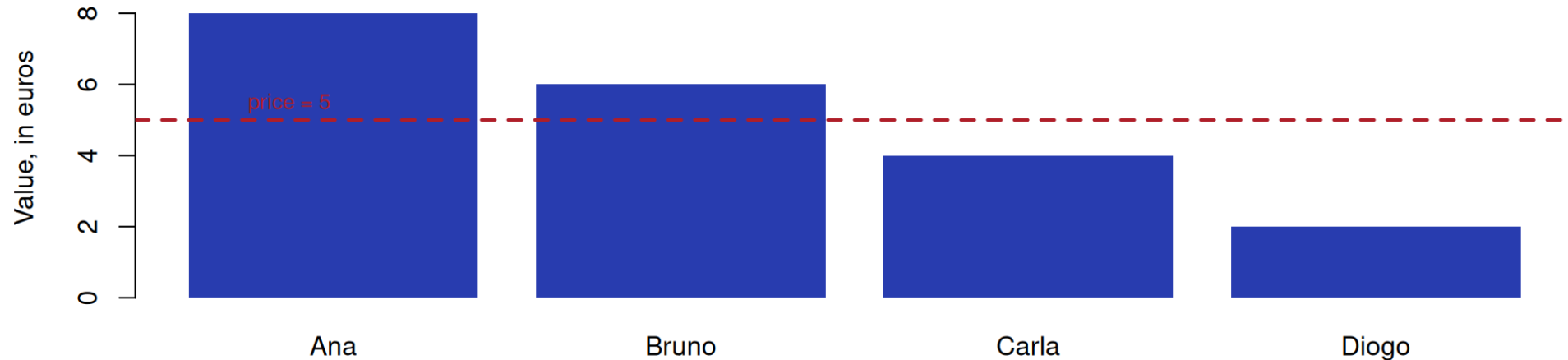
So she buys for any $p \leq v$, and the **highest** price at which she still buys is exactly v .

Reservation price of a buyer: the highest price at which they are still willing to buy. It equals their value v .

Value is what the good gives you. Price is what the market asks. The reservation price is where the two just meet. 

Four Buyers, Four Values

Ana values it at 8 euros, Bruno at 6, Carla at 4, Diogo at 2.



At a price of 5, whoever values it **above** 5 buys: Ana and Bruno. Lower the price and Carla joins, then Diogo.



The Demand Curve Emerges

Sort the reservation prices from highest to lowest and you have just drawn the **demand curve** again, from the other direction.

Each step down is one more buyer coming in. With many buyers the steps shrink and the staircase becomes a **downward sloping line**.

Demand is, at bottom, the sorted list of **willingness to pay**.
Nothing more mysterious than that. 

 Source: OpenStax *Micro* 3e, [§3.1](#), “Demand for Goods and Services”. The book builds the curve from a schedule for gasoline.

Two Ways to Read the Same Curve

Take $P = 10 - Q$.

Horizontally, at a given price: how much is bought? $Q = 10 - P$.
This is the **quantity demanded**.

Vertically, at a given quantity: what is the value of the **last** unit sold? $P = 10 - Q$. This is the **marginal willingness to pay**.

The vertical reading is the one that gives us surplus, and in a moment elasticity. Keep it. 

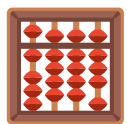
Consumer Surplus

Consumer surplus (CS): what buyers value **above** what they pay, summed over every buyer who trades.

For one buyer, $v - p$. For the market, the area **below the demand curve and above the price**.

Ana, who valued it at 8, pays 5 and walks away 3 euros better off.
Diogo, who valued it at 2, does not trade and gets nothing. 

 Source: OpenStax *Micro* 3e, [§3.5](#), “Consumer Surplus, Producer Surplus, Social Surplus”.



Consumer Surplus Is an Integral

The vertical reading gives the value of the last unit, $P(q)$. Sum value over units, subtract what was paid:

$$CS = \int_0^Q P(q) dq - pQ$$

With $P = 10 - q$ and $p = 5$, so $Q = 5$: $\int_0^5 (10 - q) dq - 25 = 37.5 - 25 = 12.5$. The triangle, without the geometry. ✓

Do not read too much into the number. Adding euros across people is a **choice** we are making here, and session 2 already warned you that utility itself is not comparable. ⚠

? Multiple Choice 1

A buyer's reservation price:

- A. Falls whenever the market price falls.
 - B. Is the price the buyer would most like to pay.
 - C. Equals the value of the good to that buyer.
 - D. Is the same for every buyer in the market.
-  **C.** The value is one number and does not move with the price. The reservation price is that value read as a price: the highest price at which buying is still worth it. What the buyer would *like* to pay is zero.

? Multiple Choice 2

Market demand is obtained from individual demands by:

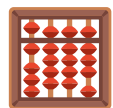
A. Adding quantities at each given price.

B. Adding prices at each given quantity.

C. Averaging the individual demand curves.

D. Taking the demand of the consumer with the highest income.

☒ **A.** At a price, each consumer names a quantity, and the market quantity is their sum. That is why the aggregation is **horizontal**. Adding prices vertically is a different operation entirely, and it belongs to public goods.



Numerical Question

Two consumers. Ana demands $q_A = 12 - 2p$ and Bruno $q_B = 8 - p$, each wherever that is positive and **zero otherwise**.

- a) Find each consumer's **choke price**, the price at which they stop buying.
- b) Write market demand as a piecewise function of p , with its ranges.
- c) Where is the kink, and why is the curve steeper above it?
- d) At $p = 3$, compute consumer surplus for Ana, and then in total.



Solution

a) Set each to zero: $12 - 2p = 0 \Rightarrow p_A = 6$, and $8 - p = 0 \Rightarrow p_B = 8$. These are the only prices where **who is in the market** changes, so they are the points the piecewise function has to break at.

b) Horizontal sum over whoever is still buying:

$$Q(p) = \begin{cases} 20 - 3p, & 0 \leq p \leq 6 \quad (\text{both}) \\ 8 - p, & 6 < p \leq 8 \quad (\text{Bruno alone}) \\ 0, & p > 8 \end{cases}$$

c) At $p = 6$, where **Ana leaves**. Below it a one euro rise costs the market 3 units; above it only 1, because Ana is no longer there to lose any. Drawn the usual way, with price on the vertical axis, losing a buyer makes the curve **steeper**. 

d) The choke prices from **a)** are the heights of the triangles. Ana buys $12 - 6 = 6$, so $CS_A = \frac{1}{2} \times 6 \times (6 - 3) = 9$ euros.

Bruno buys $8 - 3 = 5$: $CS_B = \frac{1}{2} \times 5 \times (8 - 3) = 12.5$. Total = 21.5 euros. 

Part 2 · Price Elasticity



The Point Definition

Price elasticity of demand: the percentage change in quantity over the percentage change in price.

$$\varepsilon = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

It is the derivative dQ/dP **normalized** by P/Q , so that it does not depend on units. The sign is negative; we work with $|\varepsilon|$.

Equivalently $\varepsilon = d \ln Q / d \ln P$. That form is why elasticities are estimated in logs. 



The Three Regimes

$|\varepsilon| > 1$: **elastic**. Quantity responds more than proportionally.

$|\varepsilon| < 1$: **inelastic**. Quantity responds less than proportionally.

$|\varepsilon| = 1$: **unit elastic**.



The Polar Cases



Perfectly inelastic ($\varepsilon = 0$): quantity is fixed, the curve is **vertical**. Example: a life-saving drug.

— **Perfectly elastic** ($|\varepsilon| \rightarrow \infty$): at any price above some value, demand vanishes. The curve is **horizontal**.

It is exactly what a single firm faces under perfect competition, which is why that firm is a price taker in session 4. 

 Source: OpenStax *Micro* 3e, §5.2 “Polar Cases of Elasticity and Constant Elasticity”.



Along a Linear Demand, Elasticity Varies

Take $Q = a - bP$. Then $dQ/dP = -b$ and

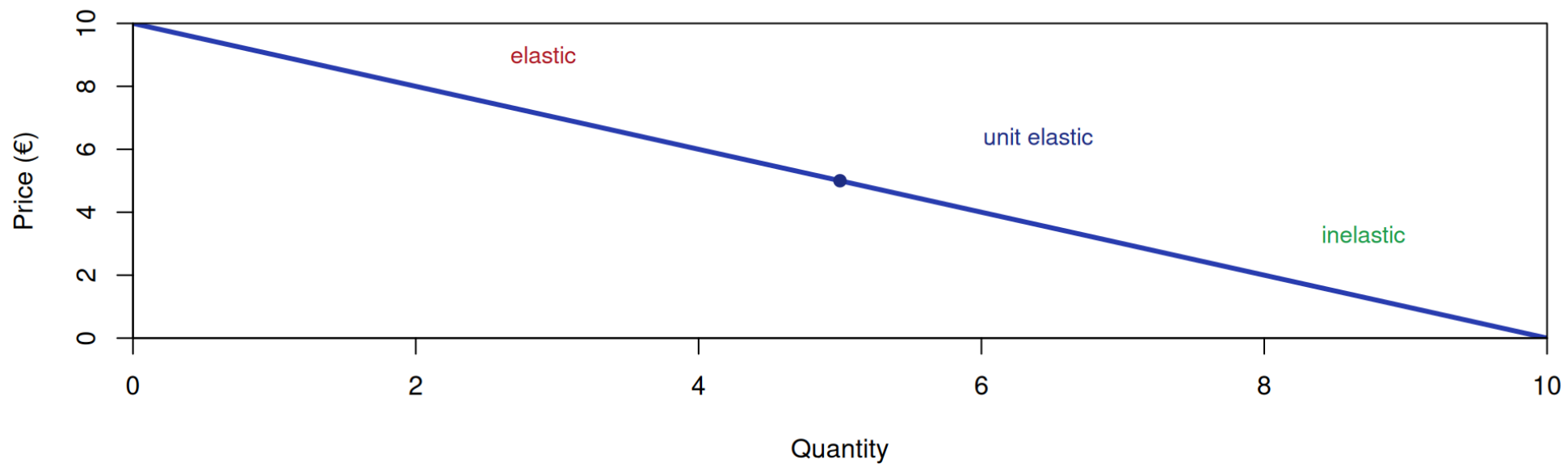
$$\varepsilon = -b \cdot \frac{P}{Q} = -b \cdot \frac{P}{a - bP}.$$

At the top of the curve (high P) it is **elastic**; at the bottom (low P) it is **inelastic**; in the middle it is **unit elastic**.

So “this good is elastic” is a statement about a **point**, not about a curve. The slope is constant, the elasticity varies along it. ⚠️



Seeing the Variation





Point or Arc?

Between two observed points, the percentage change depends on which one you start from. The **midpoint** (arc) formula avoids that:

$$\varepsilon_{arc} = \frac{\Delta Q}{(Q_1 + Q_2)/2} \bigg/ \frac{\Delta P}{(P_1 + P_2)/2}$$

It is a discrete approximation to the same object. We use the **point** definition throughout, because you already have calculus and the arc formula is just what you do when you only have two data rows. 

 OpenStax *Micro* 3e, §5.1 works with the midpoint formula, which is why its numbers differ from ours. It also prints estimated elasticities for real goods.



What Makes Demand Elastic



Substitutes. The more of them, the more elastic. This is the dominant factor.



Share of the budget. Salt is inelastic partly because nobody notices the price.



Time. Given a year, people switch heating systems; given a day, they do not. Long run elasticities are always larger.



How the market is defined. Demand for “coffee” is inelastic; for “coffee from this one brand”, very elastic. The narrower the definition, the more elastic.

? Multiple Choice 3

Along a linear demand $Q = a - bP$, elasticity in absolute value:

A. Is constant along the curve, and equal to b .

B. Rises as the price falls.

C. Is undefined except at the midpoint.

D. Falls as the price falls.

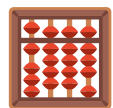
✓ **D.** $|\varepsilon| = bP/(a - bP)$ is increasing in P , so lower prices sit in the inelastic region. Option A confuses the **slope**, which is constant, with the **elasticity**, which is not.

? Multiple Choice 4

Demand is $Q = AP^{-2}$. A 1 percent rise in the price changes quantity by roughly:

- A. 2 percent up.
- B. 2 percent down.
- C. 0.5 percent down.
- D. It depends on the price you start from.

✓ **B.** $\varepsilon = \frac{dQ}{dP} \frac{P}{Q} = -2AP^{-3} \cdot \frac{P}{AP^{-2}} = -2$ at every price, so quantity falls by about twice the percentage rise in the price, from **any** starting point. Option D would be right for a linear demand and is wrong here. A demand with the same elasticity at every price has a name, which Part 3 gives it.



Numerical Question

Demand $Q = 100 - 4P$, with P in euros.

- a) Find Q at $P = 10$, and the elasticity there.
- b) Say whether demand is elastic or inelastic at that point.
- c) At which price is demand unit elastic?
- d) At which price is the choke point, where quantity hits zero?



Solution

a) $Q = 100 - 40 = 60$, and $\varepsilon = \frac{dQ}{dP} \frac{P}{Q} = -4 \times \frac{10}{60} \approx -0.67$.

b) $|\varepsilon| = 0.67 < 1$: **inelastic** at that point.

c) $|\varepsilon| = 1$ requires $4P = 100 - 4P$, so $P = 12.5$ euros and $Q = 50$. Note it is the **midpoint** of a linear demand, always.

d) $Q = 0$ at $P = 25$ euros, where $|\varepsilon| \rightarrow \infty$. At the other end, $P = 0$, elasticity is zero. The whole range is travelled along one straight line. 

Part 3 · Revenue, and the Other Elasticities

Elasticity and Revenue

Revenue is $R = P \cdot Q$. Differentiating with respect to the price,

$$\frac{dR}{dP} = Q + P \frac{dQ}{dP} = Q (1 + \varepsilon).$$

If $|\varepsilon| < 1$: raising the price **increases** revenue. If $|\varepsilon| > 1$: it **reduces** it. Revenue is maximized when $|\varepsilon| = 1$.

This tells a firm which way to move the price. 

 Source: OpenStax *Micro* 3e, §5.3, "Does Raising Price Bring in More Revenue?"



The Link to Marginal Revenue

Now differentiate with respect to the **quantity** instead:

$$MR = \frac{dR}{dQ} = P + Q \frac{dP}{dQ} = P \left(1 + \frac{1}{\varepsilon} \right).$$

Two readings. Selling one more unit brings in the price, **minus** what you gave up by cutting the price on everything else.

And $MR < P$ whenever demand slopes down. Remember this formula: it is the centre of **market power** next session. 

Income Elasticity

How does quantity change when **income** M changes?

$$\varepsilon_M = \frac{dQ}{dM} \cdot \frac{M}{Q}.$$

$\varepsilon_M > 0$: a **normal** good. $\varepsilon_M < 0$: an **inferior** good. $\varepsilon_M > 1$: a **luxury** good. 

 Source: OpenStax *Micro* 3e, §5.4, "Income Elasticity of Demand".



Cross-Price Elasticity

How does the quantity of one good change when the price of **another** good Y changes?

$$\varepsilon_{XY} = \frac{dQ_X}{dP_Y} \cdot \frac{P_Y}{Q_X}.$$

$\varepsilon_{XY} > 0$: **substitutes** (tea and coffee). $\varepsilon_{XY} < 0$: **complements** (a car and fuel). 

 Source: OpenStax *Micro* 3e, §5.4, "Cross-Price Elasticity of Demand".



Constant Elasticity Demand

Multiple Choice 4 gave you one form: $Q = A P^{-\eta}$ has $\varepsilon = \frac{dQ}{dP} \frac{P}{Q} = -\eta$ at every price.

Give income and the other good's price the same treatment, and write the exponents as the elasticities themselves:

$$Q = A P^{\varepsilon} M^{\varepsilon_M} P_Y^{\varepsilon_{XY}}$$

Take logs: $\ln Q = \ln A + \varepsilon \ln P + \varepsilon_M \ln M + \varepsilon_{XY} \ln P_Y$. Each elasticity is the **slope of a log-log regression**, and it is constant.

Which is exactly how elasticities are estimated in practice, and why you will see demand models written in logs. 

All Three, Back in the Consumer's Problem

Session 2 solved Cobb-Douglas: $x^* = \frac{\alpha M}{p_x}$. Read the three elasticities straight off it.

$\varepsilon = -1$, exactly unit elastic **everywhere**. $\varepsilon_M = 1$, so normal and never a luxury. $\varepsilon_{XY} = 0$, so neither substitute nor complement.

Those are three restrictions the functional form imposes: it is the log-log form with the exponents frozen at -1 , 1 and 0 . Cobb-Douglas is convenient and **narrow**, and a demand system estimated on real data will reject all three.

A functional form decides your answers before you see the data. ⚠️

The log-log form frees all three exponents. Doing the same to the utility function is the next step. 🙌



What People Actually Use

The fix is always the same in shape: add parameters until the elasticities are things you **estimate** rather than things the algebra handed you.

🪚 **CES**, the cheap step up. One extra parameter, the elasticity of substitution σ . Cobb-Douglas is exactly the $\sigma = 1$ case. Still one number for every pair of goods, but no longer *forced* to be one.



Sensitivity Analysis

For anyone heading into finance, elasticity is **sensitivity analysis** under another name.

How much does demand for the firm's product change if a competitor cuts its price by 5 percent? That is cross-price elasticity.

How far does demand fall in a recession? That is income elasticity. The same formulas, applied to revenue models. 

? Multiple Choice 5

A firm currently sells where demand is **inelastic**. To raise revenue it should:

- A. Cut the price, because quantity will respond strongly.
- B. Raise the price, because quantity falls less than proportionally.
- C. Leave the price alone, since revenue is already at a maximum.
- D. Cut the price, because revenue always rises when the price falls.

✓ **B.** With $|\varepsilon| < 1$ we have $dR/dP = Q(1 + \varepsilon) > 0$. Revenue keeps rising with the price until the unit elastic point, where $dR/dP = 0$.

? Multiple Choice 6

The cross-price elasticity between two goods is **negative**. The goods are:

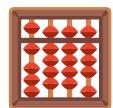
A. Complements.

B. Substitutes.

C. Independent of each other.

D. Both inferior goods.

☒ **A.** The price of one rises and demand for the other falls, so they are consumed together: complements. Note that income elasticity, not cross-price elasticity, is what classifies a good as inferior.



Numerical Question

Demand for a product is estimated as

$$\ln Q = 6 - 1.5 \ln P + 0.8 \ln M + 0.4 \ln P_Y.$$

- a) State the three elasticities.
- b) Is the good normal, inferior or a luxury?
- c) Is Y a substitute or a complement?
- d) The firm raises its price by 4 percent. What happens to revenue, roughly?



Solution

- a) In a log-log form the coefficients **are** the elasticities: $\varepsilon = -1.5$, $\varepsilon_M = 0.8$, $\varepsilon_{XY} = 0.4$.
- b) $0 < \varepsilon_M < 1$: **normal**, but not a luxury. Demand grows with income, more slowly than income does.
- c) $\varepsilon_{XY} > 0$: a **substitute**. Y becoming dearer sends buyers this way.
- d) $|\varepsilon| = 1.5 > 1$, so revenue **falls**. Quantity drops about $1.5 \times 4 = 6$ percent against a 4 percent price rise, for a net change near -2 percent. 

Wrap-Up

What to Take From This Session

✚ Market demand is individual demands **added horizontally**. It slopes down because each consumer's does.

♥ **Consumer surplus** is value above price: the area under demand and above the price, or an integral if you prefer.

📐 Elasticity is a **normalized derivative**, $d \ln Q / d \ln P$. Along a linear demand it runs from elastic at the top to inelastic at the bottom.

💰 $dR/dP = Q(1 + \varepsilon)$ tells a firm which way to move the price, and $MR = P(1 + 1/\varepsilon)$ is the same fact on the quantity side.

💰 Income and cross-price elasticities classify goods: normal or inferior, substitute or complement.

Next Session

Production, Costs and Market Structures.

We have the buyers. Next we build the **seller**: where costs come from, why supply is marginal cost, and what changes when one firm faces the whole demand curve. 

See you next week. 