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Principles of Economics and Management I

Session 2 · Consumer Choice

Introduction

Where We Came From

Session 1 gave the whole logic of a rational choice: **surplus** $B - C$, and the **opportunity cost** that decides between options.

It also showed that a slope can *be* an opportunity cost. That was the MRT of the production frontier, for a whole economy. 

Today the same reading returns on the consumer's side, and we write the problem down properly and **solve** it. You already have the mathematics; we are only supplying the economics.



Today's Map

1. 📜 What we assume about preferences, and what utility is for
2. ⚠️ Why the **number** utility takes carries no information
3. 📈 Indifference curves, and what their slope is
4. 💶 The budget set, and what moves it
5. 📊 $\max U$ subject to the constraint, straight to the **Lagrangian**
6. 💡 λ as a **shadow price**
7. 📈 The solution as a function of prices and income: **Marshallian demand**
8. ➕ A second functional form, and why the shape decides the answers

We finish holding a demand function. Next session turns it into a market. 🔧

Part 1 · Preferences and Utility



What We Assume About Preferences

Before any function, four assumptions on the ranking $\succsim$ over bundles:

Complete: for any two bundles, the consumer can say which they prefer, or that they are indifferent.

Transitive: if $a \succsim b$ and $b \succsim c$ then $a \succsim c$. Without this there is no “best” bundle to find.

Monotone: more is better.

Convex: averages are at least as good as extremes.



What We Assume About Preferences

Complete and **transitive** are what let us speak of a best bundle at all.

Completeness rules out “I cannot say”. Put any two bundles in front of the consumer and there is an answer.

Transitivity rules out cycles. Suppose $a \succ b$, $b \succ c$, and $c \succ a$. 

Then I sell you a for your c plus a cent, then b for your a plus a cent, then c for your b plus a cent. You are holding what you started with, three cents lighter, and I can run it again. 

A consumer with intransitive preferences is a machine for making me rich. So we assume it away.



What We Assume About Preferences

Monotone and **convex** are about shape, and both can fail.

Monotone says more is better. It fails for a **bad**: nobody wants more noise, more pollution, more risk. 📢

It also fails at satiation. The fifteenth coffee may be worth less than nothing. ☕

Convex says averages are at least as good as extremes. Half a coffee and half a cake beats two coffees or two cakes, for most people.

It fails when two goods do not mix. Half a portion of rice with half a portion of breakfast cereal is worse than a full portion of either. 🍚

Utility

Given those assumptions, we can represent $\succsim$ by a **utility** function $U(x, y)$.

It is a **representation**, nothing more: $U(a) \geq U(b)$ means exactly $a \succsim b$.

Write $U_x \equiv \partial U / \partial x$ throughout. Monotonicity is $U_x > 0$ and $U_y > 0$; diminishing marginal utility is $U_{xx} < 0$ and $U_{yy} < 0$: the tenth coffee is worth less than the first. ☕

Careful with those two: $U_{xx} < 0$ is a property of the **function**, while convexity of preferences is a property of its **level sets**: the set of bundles at least as good as a given one is convex. Only the second is a statement about preferences, for the reason on the next slide. ⚠

 Source: OpenStax *Micro* 3e, §6.1 "Consumption Choices", "Total Utility and Diminishing Marginal Utility".



Utility Is Ordinal, Not Cardinal

This point costs one slide and saves a great deal of confusion later.

If f is strictly increasing, U and $f(U)$ describe **exactly the same preferences**. $U = xy$ and $V = \ln x + \ln y$ are the same consumer.

So the **number** utility takes is meaningless. Only the **ranking** survives, and with it any ratio like U_x/U_y .

“Ana has 40 utils and Bruno 20, so Ana is twice as happy” is not a statement economics can make. 



Utility Is Ordinal, Not Cardinal

Take $U = xy$ and $V = \ln x + \ln y$, and three bundles:

Bundle	$U = xy$	$V = \ln x + \ln y$
(1, 4)	4	1.386
(2, 2)	4	1.386
(3, 3)	9	2.197

The numbers are different. The ranking is identical, and the first two bundles are still tied.

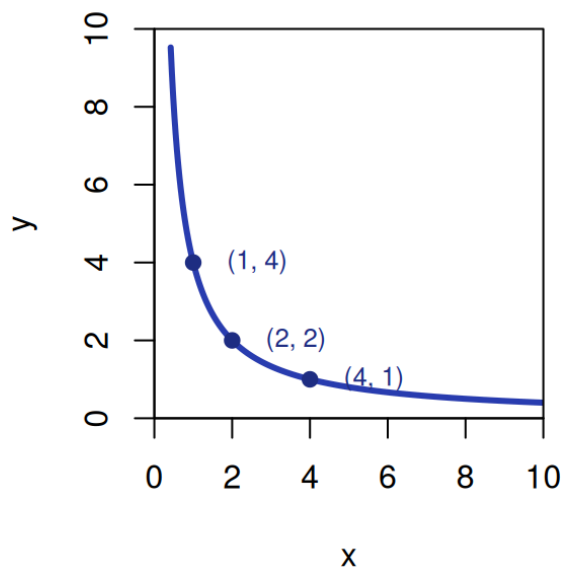
And the ratio survives: $V_x/V_y = (1/x)/(1/y) = y/x$, which is what U gives. Everything we are about to do runs on that ratio. 



Indifference Curves

If only the ranking survives, draw the ranking. Fix a number and collect every bundle that delivers it.

With $U = xy$, the bundles worth exactly 4:

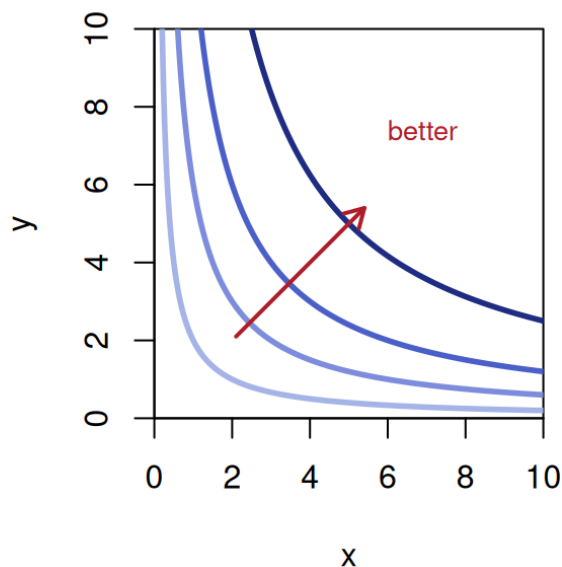


Every bundle on the curve is worth the same to this consumer. That is what indifferent means.



Indifference Curves

Change the number and you get another curve. Do it repeatedly and you have the **map**:



Curves further out are better, because more is better. They never cross: one bundle cannot sit on two different rankings. And $V = \ln x + \ln y$ draws **these same curves**, with different numbers on them. ↺



The Slope of an Indifference Curve

Along one curve, U does not change. Take the total differential and set it to zero:

$$U_x dx + U_y dy = 0 \quad \implies \quad \frac{dy}{dx} = -\frac{U_x}{U_y}$$

So the ratio U_x/U_y **is** the slope of the curve you just drew, up to the minus sign. 

It answers one question: how much y will this consumer give up for one more x , and stay exactly as well off?

Hold that question. In Part 2 the **market** answers the same one, with p_x/p_y . 



Two Shapes to Keep in Mind

Two limiting cases are convex, but only just:

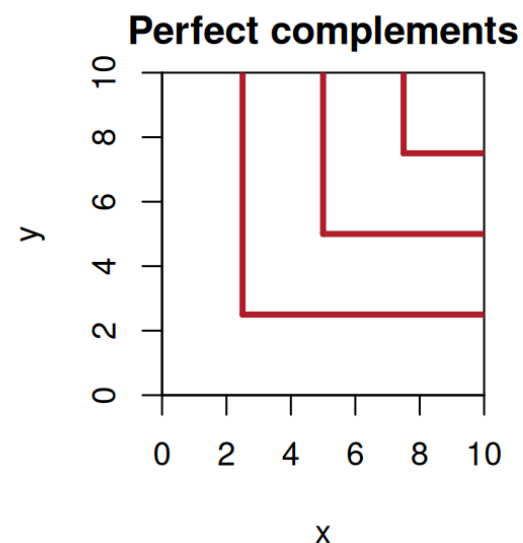
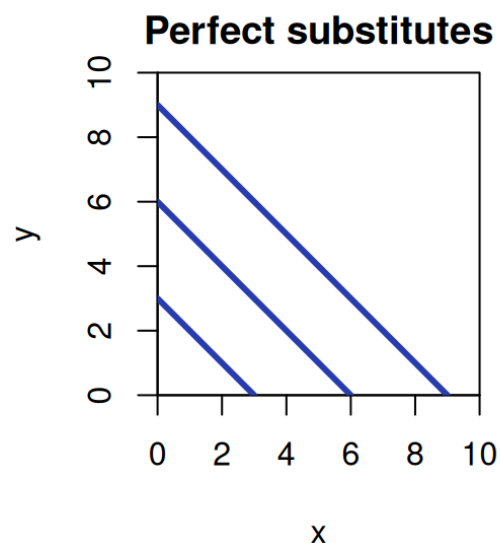
 **Perfect substitutes:** $U = ax + by$. Here $U_x/U_y = a/b$ at every bundle, so the consumer gives up y for x at the same exchange rate whatever they already hold.

 **Perfect complements:** $U = \min\{ax, by\}$. The goods are used in fixed proportions, and more x without more y adds nothing. At the corner of each curve, U_x/U_y does not exist.

We come back to both in Part 2, once there are prices. ⚠



Two Shapes to Keep in Mind



Straight lines on the left: the slope is $-a/b$ everywhere, so the trade-off never changes along a curve.

Right angles on the right: starting from the corner, extra x alone or extra y alone leaves the consumer on the same curve. There is no slope at the corner at all. 

? Multiple Choice 1

$U = xy$ is replaced by $V = \ln x + \ln y$. The indifference curves of V :

A. Are the same curves as those of U , with different numbers on them.

B. Are different, because the utility numbers are smaller.

C. Are straight lines, because V is a sum.

D. Cannot be drawn where V is negative, that is where $xy < 1$.

✓ **A.** $V = \ln U$ is a strictly increasing transformation, so it ranks every pair of bundles the same way and represents the same preferences. Check the ratio:
 $V_x/V_y = (1/x)/(1/y) = y/x$, exactly what U gives.

Discussion Question

For $U(x, y) = x^\alpha y^{1-\alpha}$, with $0 < \alpha < 1$:

Show that $\frac{U_x}{U_y} = \frac{\alpha}{1-\alpha} \cdot \frac{y}{x}$, and say what it depends on.

$$U_x = \alpha x^{\alpha-1} y^{1-\alpha} \text{ and } U_y = (1 - \alpha) x^\alpha y^{-\alpha}.$$

Dividing, $\frac{U_x}{U_y} = \frac{\alpha}{1-\alpha} \cdot \frac{y}{x}$. It depends only on the **ratio** y/x , not on the level of utility. 

Part 2 · The Consumer's Problem

The Budget Set

With income M and prices p_x, p_y , the consumer can afford any bundle with

$$p_x x + p_y y \leq M$$

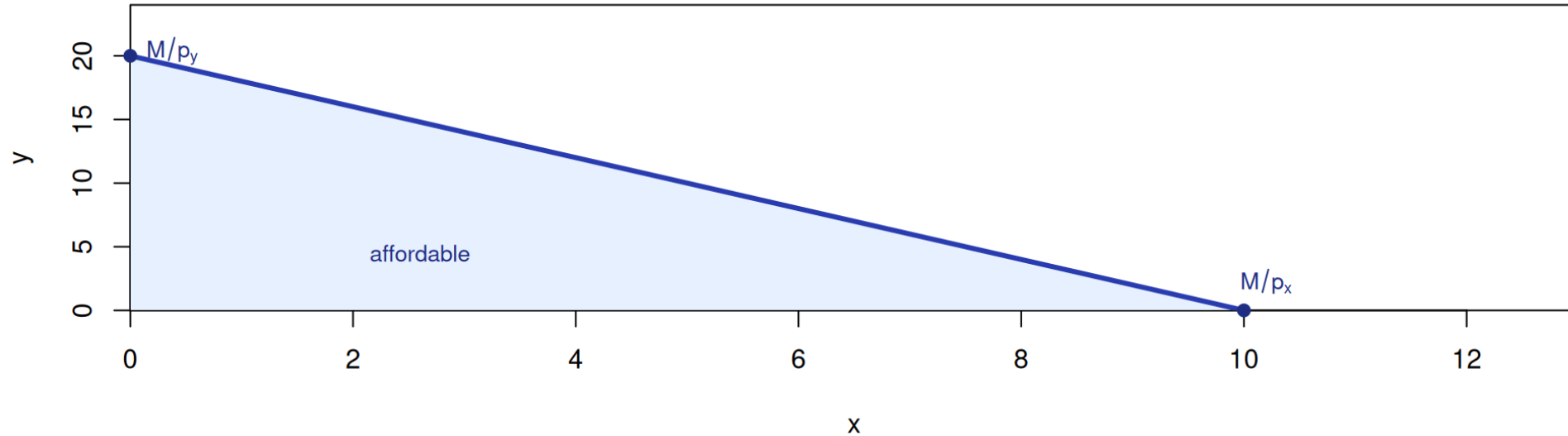
The boundary is the **budget constraint**, a line of slope $-p_x/p_y$: the rate at which the **market** lets you trade x for y . 

That rate **is** an opportunity cost, in the session 1 sense: one more x costs p_x/p_y units of y . The *MRT* said what the **economy** gives up; p_x/p_y says what the **consumer** does. 

Monotone preferences mean the consumer never stops inside the set, so the constraint binds with **equality**. That is what lets us use a Lagrangian.



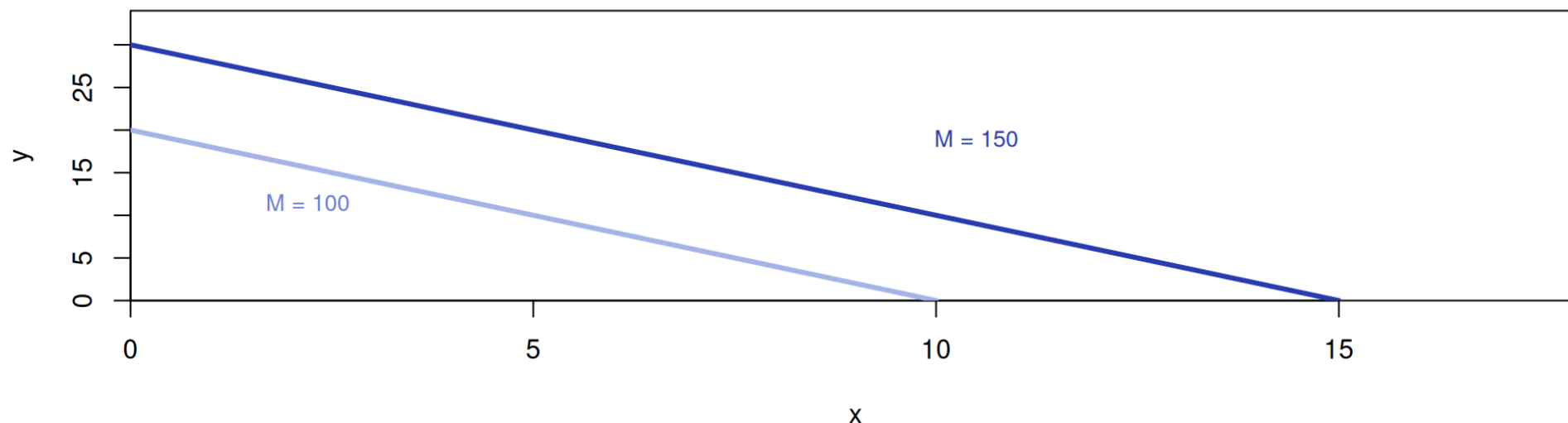
The Budget Set



With $M = 100$, $p_x = 10$ and $p_y = 5$: at most 10 of x , at most 20 of y , and the line between them has slope $-p_x/p_y = -2$.

The Budget Set

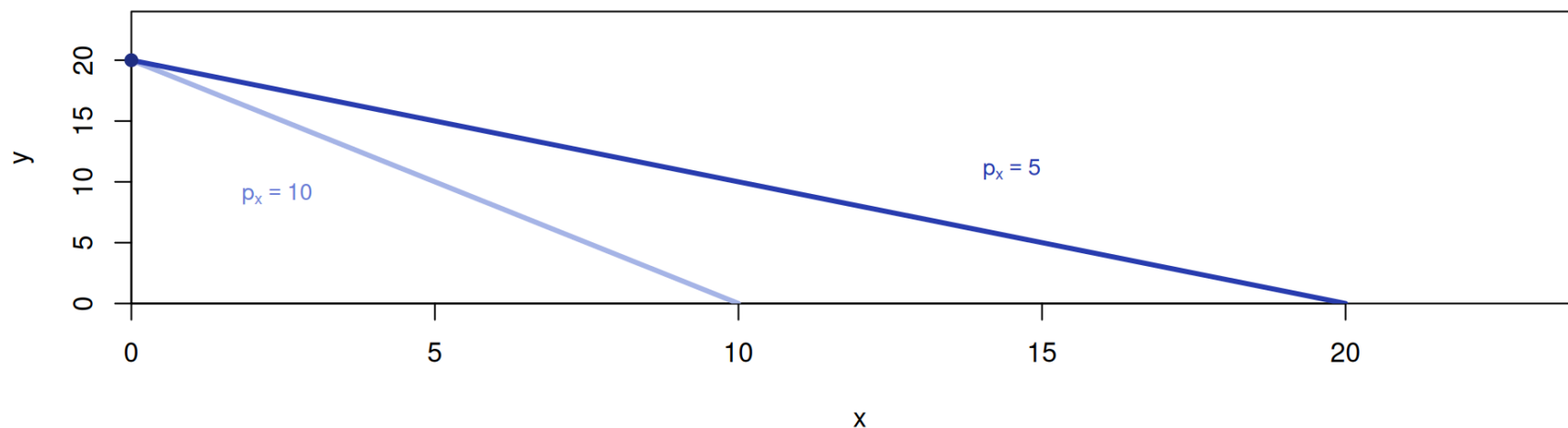
Raise income to 150, prices unchanged:



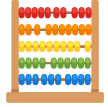
The line moves **out, parallel**. No price changed, so the rate at which the market lets you swap x for y is what it was. You can simply afford more of both.

The Budget Set

Now hold income at 100 and let p_x fall from 10 to 5:



The line **rotates** about the y intercept. Spending everything on y buys exactly what it bought before, and the slope went from -2 to -1 : x is now cheaper in terms of y .



The Problem, Written Down

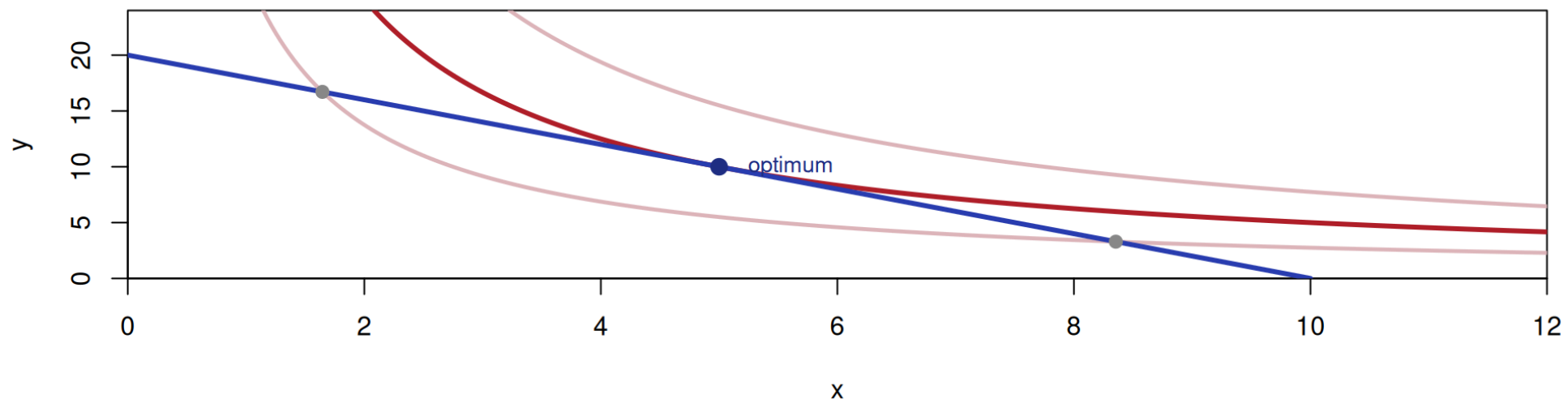
Everything so far collapses into one line:

$$\max_{x, y} U(x, y) \quad \text{subject to} \quad p_x x + p_y y = M$$

A smooth objective, one linear constraint. You have solved this shape many times. With **strictly convex preferences** the first order conditions are **sufficient**, and the solution is unique, so there is no second order check to do.

So we go straight to the **Lagrangian**, and spend the time on what the solution *means* rather than on how to find it.

The Solution, Drawn



The best affordable bundle is where an indifference curve just **touches** the budget line.

The Solution, Drawn

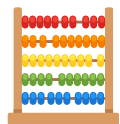
The two grey bundles in that figure are affordable too, and both sit on a **lower** curve.

Between them the consumer can do better, and the best they can do is the touching point. 

At that point the two slopes agree:

$$-\frac{U_x}{U_y} = -\frac{p_x}{p_y}$$

The rate the consumer is willing to swap at equals the rate the market swaps at. Which is the equation we are about to get out of the Lagrangian. 



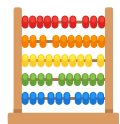
The Lagrangian

$$\mathcal{L} = U(x, y) + \lambda (M - p_x x - p_y y)$$

First order conditions:

$$U_x = \lambda p_x, \quad U_y = \lambda p_y, \quad p_x x + p_y y = M$$

Three equations, three unknowns (x, y, λ) . Solving them is the whole problem.



The Lagrangian

Do it once with numbers. $U = xy$, $p_x = 10$, $p_y = 5$, $M = 100$:

$$\mathcal{L} = xy + \lambda(100 - 10x - 5y)$$

$$\mathcal{L}_x = y - 10\lambda = 0, \quad \mathcal{L}_y = x - 5\lambda = 0, \quad 100 - 10x - 5y = 0$$

From the first two, $y = 10\lambda$ and $x = 5\lambda$. Put them in the third: $100 = 50\lambda + 50\lambda$, so $\lambda = 1$.

Then $x^* = 5$ and $y^* = 10$, costing $10(5) + 5(10) = 100$. ✓

Which is the bundle the figure two slides back was pointing at. 🎯



Reading the First Order Conditions

Divide the first two:

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

Rearranged the other way, $\frac{U_x}{p_x} = \frac{U_y}{p_y}$: the **marginal utility per euro** is the same in every direction.

If it were not, move one euro from the low-return good to the high-return one and gain. The condition is exactly “no free improvement left”.



Reading the First Order Conditions

Suppose it fails. At some affordable bundle, $U_x/p_x = 3$ and $U_y/p_y = 1$.

Move one euro off y and onto x . You lose 1 unit of utility and gain 3. Net: **+2**. 

So that bundle was not the best one. Keep moving euros while the two sides differ.

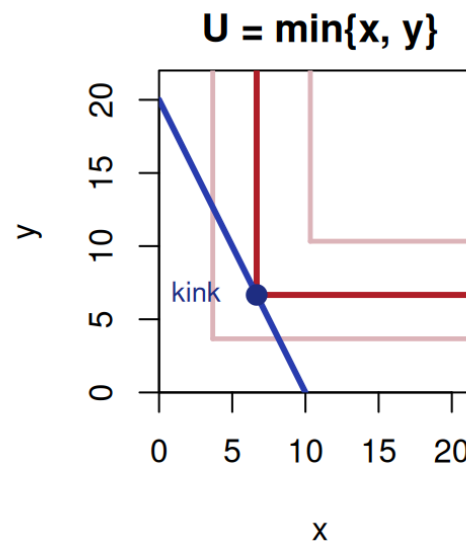
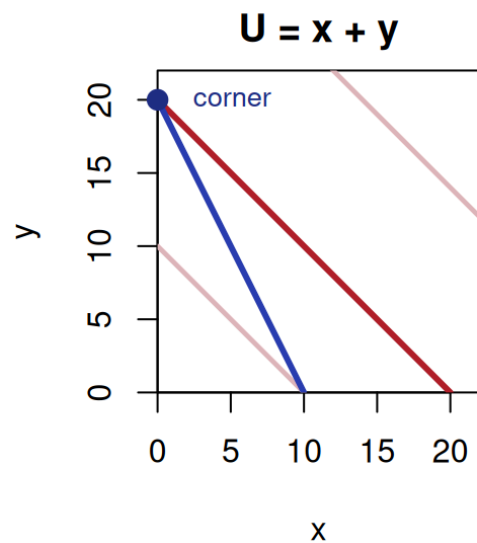
As x rises U_x falls, and as y falls U_y rises, so the two ratios close on each other. They meet exactly where the first order conditions hold, and there is nothing left to gain. 

 Source: OpenStax *Micro* 3e, §6.1, "Choosing with Marginal Utility", which reaches the same rule arithmetically.



Two Shapes to Keep in Mind

Back to the two shapes from Part 1, now with $p_x = 10$, $p_y = 5$ and $M = 100$:



Neither optimum is a point where a curve touches the budget line with the same slope.



Two Shapes to Keep in Mind

Run the euro-moving argument on $U = x + y$:

$U_x/p_x = 1/10$ and $U_y/p_y = 1/5$, at **every** bundle. Moving a euro from x to y always gains.

The two ratios never meet, so the consumer keeps moving euros until there is no x left: the **corner** $x^* = 0$, $y^* = 20$. The first order conditions do not hold with equality there.

For $U = \min\{x, y\}$, U_x/U_y does not exist at the kink, so there is nothing to set equal to p_x/p_y . The optimum comes from the kink itself, $x = y$, and the budget: $15x = 100$, so $x^* = y^* = 20/3$.

Watch for both in the problem sets. ⚠



What λ Actually Is

The optimum depends on what the consumer faces. Change p_x , p_y or M and the conditions give a different (x^*, y^*) .

So the utility reached at the optimum is also a function of prices and income:

$$U^*(p_x, p_y, M) = U(x^*(p_x, p_y, M), y^*(p_x, p_y, M))$$

Indirect utility: the utility of the best affordable bundle, as a function of prices and income.

In optimization terms, U^* is the **value function** of the consumer's problem: the maximized objective, as a function of the parameters (p_x, p_y, M) .

With $U = xy$, $p_x = 10$, $p_y = 5$ and $M = 100$ we found $x^* = 5$ and $y^* = 10$, so $U^* = 50$. Part 3 looks at $x^*(\cdot)$ itself; for λ we only need U^* .



What λ Actually Is

From any first order condition, $\lambda = U_x/p_x = U_y/p_y$.

By the envelope theorem, the derivative of U^* in M is the derivative of $\mathcal{L}$ in the explicit M only, at the optimum:

$$\frac{\partial U^*}{\partial M} = \frac{\partial \mathcal{L}}{\partial M} = \lambda > 0$$

λ is the **marginal utility of income**: what one more euro of budget is worth to this consumer. A **shadow price**.

Keep it. The same reading returns for the firm in session 4, and for the government's budget in session 12. 



What λ Actually Is

Back to $U = xy$, $p_x = 10$, $p_y = 5$, $M = 100$, where we found $\lambda = 1$ and $U^* = 5 \times 10 = 50$.

Raise income to $M = 101$ and redo it: $x^* = 5.05$, $y^* = 10.1$, so $U^* = 51.005$.

The extra euro bought **1.005** units of utility, against a predicted $\lambda = 1$. The gap is the second order term: λ is a derivative, so it is exact only in the limit. 

Now instead raise p_x to 20, at $M = 100$. Then $x^* = 2.5$, $y^* = 10$, and $\lambda = U_x/p_x = 10/20 = 0.5$.

The same euro of income is worth **half** as much, because the bundle it buys is worse. That is what makes λ a price. 

? Multiple Choice 2

In $\mathcal{L} = U(x, y) + \lambda(M - p_x x - p_y y)$, the multiplier λ is:

- A. The price ratio p_x/p_y .
 - B. The marginal utility of income.
 - C. The consumer's total utility at the optimum.
 - D. Always equal to one at an interior optimum.
-  **B.** From the first order conditions $\lambda = U_x/p_x = U_y/p_y$, the marginal utility per euro. By the envelope theorem it is exactly $\partial U^* / \partial M$.

? Multiple Choice 3

At the optimum $U_x/U_y = p_x/p_y$. Suppose instead $U_x/p_x > U_y/p_y$. The consumer should:

- A. Stop: the bundle is already optimal.
 - B. Buy less of both goods.
 - C. Move a euro from y to x , which raises utility.
 - D. Raise income, the only way to gain.
- ✓ **C.** A euro spent on x buys more utility than a euro spent on y , so shifting spending gains. Only when the two are equal is there nothing left to take.

? Multiple Choice 4

For which preferences should you **not** expect the first order conditions to hold with equality?

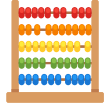
A. $U = x^{0.5}y^{0.5}$

B. $U = \ln x + 2 \ln y$

C. $U = x^{0.3}y^{0.7}$

D. $U = \min\{2x, y\}$

✓ **D.** Perfect complements have a kink, so U is not differentiable there and the solution sits at the kink rather than where derivatives balance. The other three are smooth.



Numerical Question

$U(x, y) = x y$, with $p_x = 2$, $p_y = 1$ and $M = 20$ euros.

- a) Find the optimal bundle.
- b) Compute λ and interpret it.
- c) Now p_x falls to 1. Find the new bundle.
- d) What share of the budget goes to each good, and why is that no accident?



Solution

- a) From the conditions, $y/x = p_x/p_y = 2$, so $y = 2x$. In the constraint $2x + 2x = 20$, giving $x^* = 5$ and $y^* = 10$.
- b) $\lambda = U_x/p_x = y/p_x = 10/2 = 5$. One more euro of income would raise utility by about 5 units.
- c) Now $y/x = 1$, so $y = x$, and $x + x = 20$ gives $x^* = y^* = 10$. The quantity of x doubled.
- d) $p_x x^* = 10$ and $p_y y^* = 10$: **half the budget on each, both times**. Cobb-Douglas $x^\alpha y^{1-\alpha}$ always spends the constant shares α and $1 - \alpha$.

Part 3 · Marshallian Demand



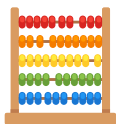
The Solution Is a Function

Solve the three conditions and the answer is a **rule**:

$$x^* = x(p_x, p_y, M), \quad y^* = y(p_x, p_y, M)$$

Marshallian demand: the optimal quantity, as a function of prices and income.

Every question about how a consumer responds to anything is a question about the derivatives of this function.



Cobb-Douglas, Solved

Take $U = x^\alpha y^{1-\alpha}$. The conditions give $\frac{\alpha}{1-\alpha} \cdot \frac{y}{x} = \frac{p_x}{p_y}$, so $p_y y = \frac{1-\alpha}{\alpha} p_x x$.

Substitute into $p_x x + p_y y = M$:

$$x^* = \frac{\alpha M}{p_x}, \quad y^* = \frac{(1 - \alpha)M}{p_y}$$

Read it off: spending on x is $p_x x^* = \alpha M$, a **constant share** of income. That is the Cobb-Douglas signature. 



What the Demand Function Says



Own price: $\frac{\partial x^*}{\partial p_x} = -\frac{\alpha M}{p_x^2} < 0$. Demand slopes down.



Income: $\frac{\partial x^*}{\partial M} = \frac{\alpha}{p_x} > 0$, a **normal** good. Where this is negative the good is **inferior**.



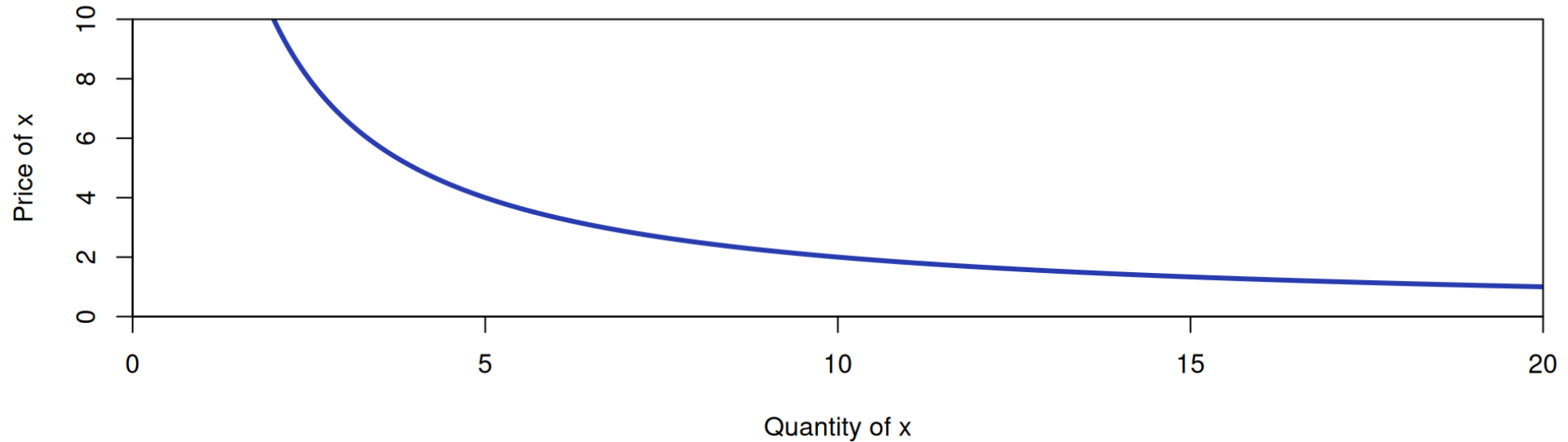
Other price: here $\frac{\partial x^*}{\partial p_y} = 0$. Cobb-Douglas is special; in general this is where substitutes and complements live.

And x^* is unchanged if all prices and income double: demand is **homogeneous of degree zero**. Only relative prices matter.



This Is the Demand Curve

Fix p_y and M , vary p_x , and plot the pairs (x^*, p_x) .

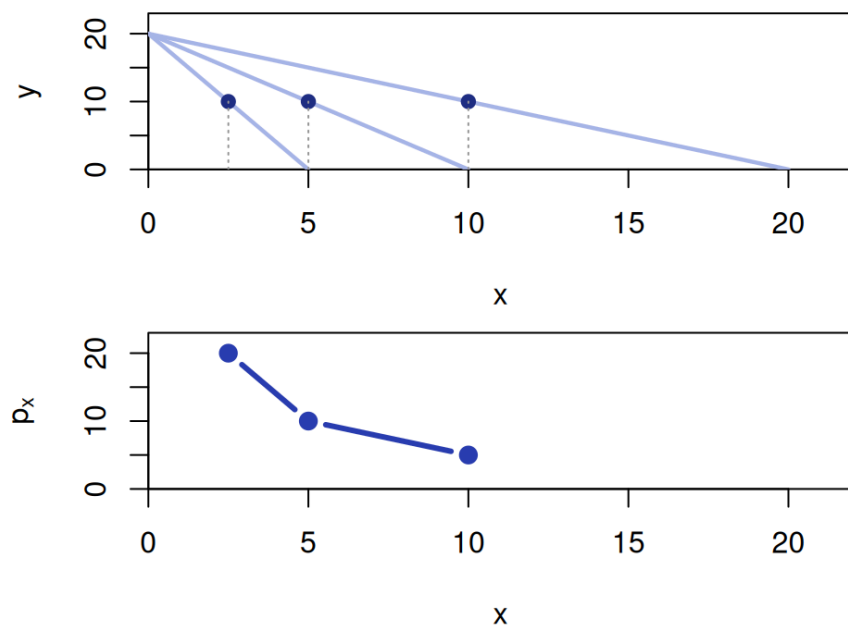


The **solution** to the problem we just solved, drawn.



This Is the Demand Curve

Where it comes from: three prices, three budget lines, three optima. 🙋



Read down: each optimum above hands one point to the panel below. Join them and that is the demand curve.

 Source: OpenStax *Micro* 3e, §6.2, "The Foundations of Demand Curves".

+ A Second Shape: Quasi-Linear

Cobb-Douglas is one functional form, and a very particular one. Here is another:

$$U(x, y) = v(x) + y$$

Utility is **linear** in y and curved only in x . Read y as “everything else”, measured in euros. 

Take $v(x) = 2\sqrt{x}$, so $U_x = 1/\sqrt{x}$ and $U_y = 1$.

The slope of an indifference curve is then $-U_x/U_y = -1/\sqrt{x}$. It depends on x alone, and not at all on y . 

+ A Second Shape: Quasi-Linear

Set that slope against the price ratio, with $p_y = 1$:

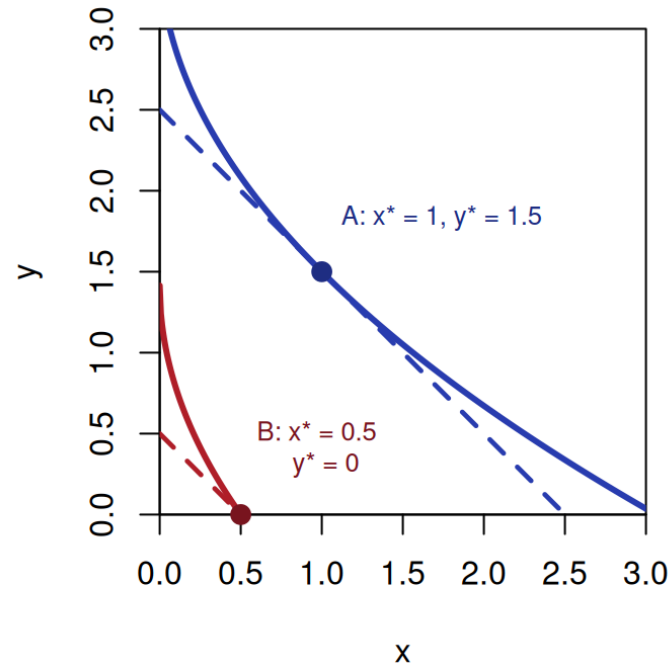
$$\frac{1}{\sqrt{x}} = p_x \quad \implies \quad x^* = \frac{1}{p_x^2}$$

Look at what is missing: M does not appear. 🔍

Demand for x does not respond to income at all. Every extra euro goes to y , and $y^* = M - 1/p_x$ is where it went.

That holds only while M is large enough to pay for x^* . Below that the consumer is at a corner, and the first order conditions stop holding with equality. ⚠️

+ A Second Shape: Quasi-Linear



One consumer, $p_x = p_y = 1$, two incomes. Blue: $M = 2.5$. Red: $M = 0.5$. Dashed lines are the budgets.

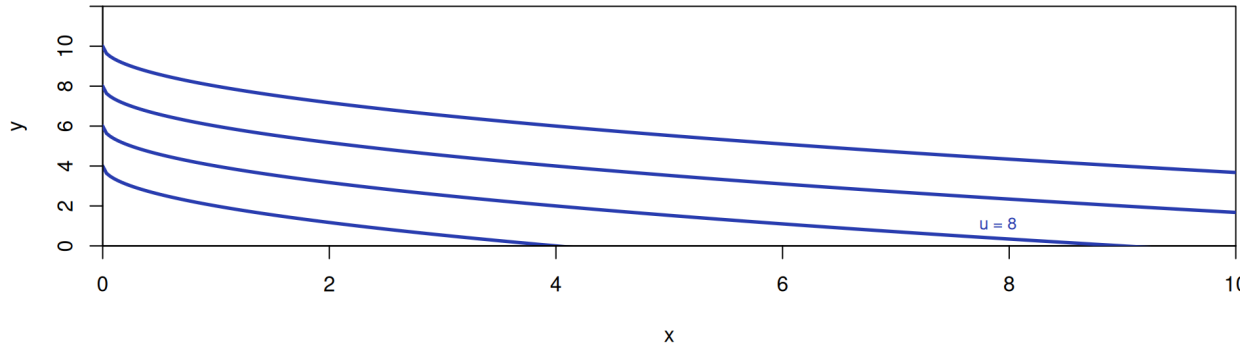
+ A Second Shape: Quasi-Linear

Whatever M is, the first order condition asks for $x^* = 1/p_x^2 = 1$, which costs 1.

A: with $M = 2.5$ that is affordable. The curve touches the budget line and the rest goes to y , $y^* = 1.5$.

B: with $M = 0.5$ it is not. So all of it goes to x : $x^* = 0.5$, $y^* = 0$. There the curve is steeper than the budget line, $1/\sqrt{0.5} \approx 1.41 > 1$, so the consumer would still give up y for more x ... but there is no y left.

+ A Second Shape: Quasi-Linear



Every curve is the same curve shifted **vertically**. So the slope at a given x is identical on all of them, which is exactly why M dropped out.

It matters in session 3: with quasi-linear utility the area under the demand curve **is** the consumer's gain in euros. With Cobb-Douglas it is an approximation. ↻



Two Forms, Side by Side

	Cobb-Douglas $x^\alpha y^{1-\alpha}$	Quasi-linear $2\sqrt{x} + y$
x^*	$\alpha M / p_x$	$1/p_x^2$
Responds to income	one for one in spending	not at all
Share of income on x	fixed at α	falls as M rises
Effect of p_y	none	none

Neither one is “the” consumer. Each is a guess about shape, and the guess decides the answers before any data arrives. Session 3 makes the same point again, with elasticities. 

? Multiple Choice 5

All prices and income double. The consumer's optimal bundle:

- A. Doubles, since income doubled.
 - B. Halves, since everything is dearer.
 - C. Is unchanged: demand is homogeneous of degree zero.
 - D. Cannot be determined without knowing U .
- ✓ **C.** The budget constraint is unchanged when both sides scale, and so is the price ratio. Only **relative** prices and **real** income matter, which is why money illusion is a mistake and not a preference.

? Multiple Choice 6

For $U = x^\alpha y^{1-\alpha}$, income rises 10% with prices fixed. Spending on x :

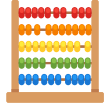
A. Rises by less than 10%, since some goes to y .

B. Rises by exactly 10%.

C. Is unchanged.

D. Depends on p_x and p_y .

✓ **B.** Spending on x is αM , so it moves one for one with income. The **share** α stays fixed, which is what makes Cobb-Douglas convenient and restrictive at once.



Numerical Question

$U(x, y) = x^{0.25}y^{0.75}$, with income M and prices p_x, p_y .

- a) Derive the Marshallian demands x^* and y^* .
- b) What share of income goes to each good?
- c) Is x normal or inferior? Show it.
- d) With $M = 100$, $p_x = 5$, $p_y = 3$, compute the bundle.



Solution

- a) $x^* = \frac{0.25 M}{p_x}$ and $y^* = \frac{0.75 M}{p_y}$, the Cobb-Douglas rule with $\alpha = 0.25$.
- b) $p_x x^* = 0.25M$ and $p_y y^* = 0.75M$: a quarter and three quarters, whatever the prices.
- c) $\partial x^* / \partial M = 0.25 / p_x > 0$, so x is **normal**. Cobb-Douglas cannot produce an inferior good.
- d) $x^* = 25/5 = 5$ and $y^* = 75/3 = 25$. Check: $5(5) + 3(25) = 100$. 

Wrap-Up

What to Take From This Session

 **Utility** is a way of writing a preference ranking down. Any increasing transformation of it is the same consumer.

 The problem is $\max U$ subject to a linear constraint. The **Lagrangian** solves it, and the first order conditions say marginal utility per euro is equal everywhere.

 λ is the **marginal utility of income**, a shadow price. It returns for the firm and for the government.

 The solution is a **function**, the Marshallian demand $x(p_x, p_y, M)$. The demand curve is that function drawn.

 Corners and kinks break the first order conditions. Perfect substitutes and perfect complements are the two you will meet.

 **Cobb-Douglas** and **quasi-linear** answer the same problem differently. The functional form is an assumption, not a finding.

Next Session

Demand and Elasticity.

From one consumer to a **market**, and then the question that decides revenue: how sharply does quantity respond to price? 